"Teach A Level Maths" Vol. 1: AS Core Modules

Demo Disc

The following slides show one of the 51 presentations that cover the AS Mathematics core modules C1 and C2.

AS Mathematics

26: Definite Integration and Areas

Definite Integration and Areas

Module C1

AQA

Module C2

Edexcel

MEI/OCR

OCR

Definite Integration and Areas

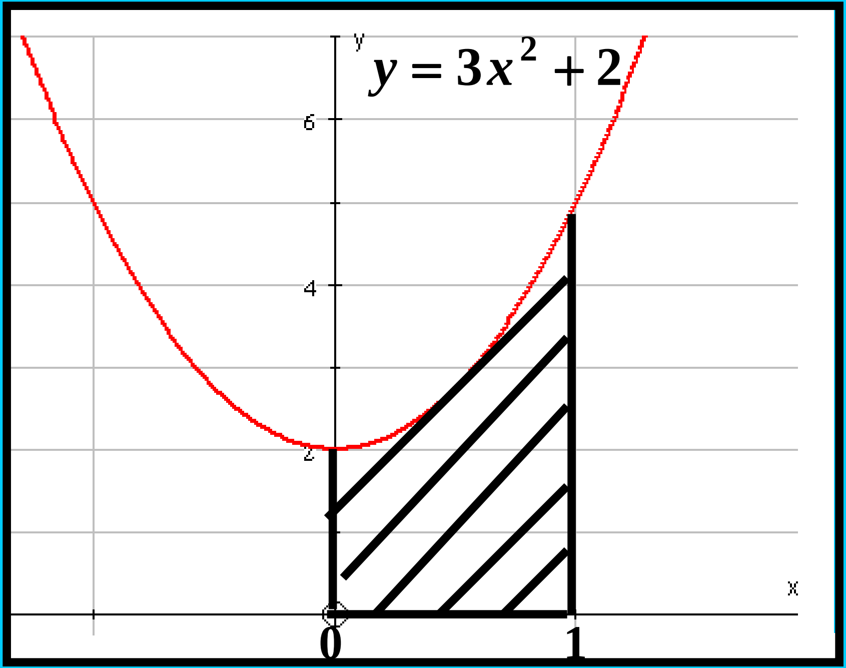
Areas

Definite integration results in a **value.**

It can be used to find an area bounded, in part, by a curve

e.g. $\int_0^1 3x^2 + 2\,dx$ gives the area shaded on the graph

The limits of integration . . .



Definite Integration and Areas

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The limits of integration . . .

. . . give the boundaries of the area.

$x = 0$ is the lower limit

(the left hand boundary)

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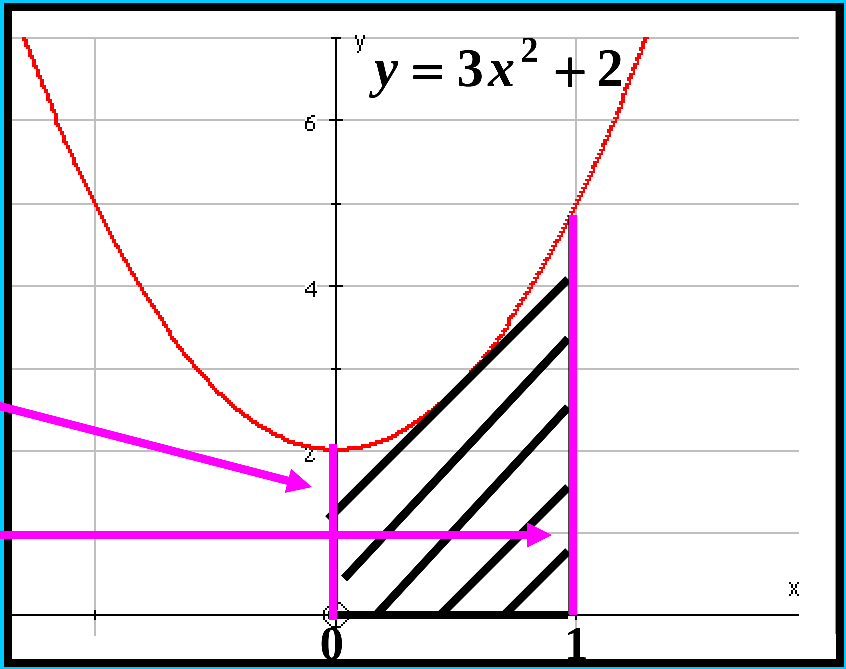


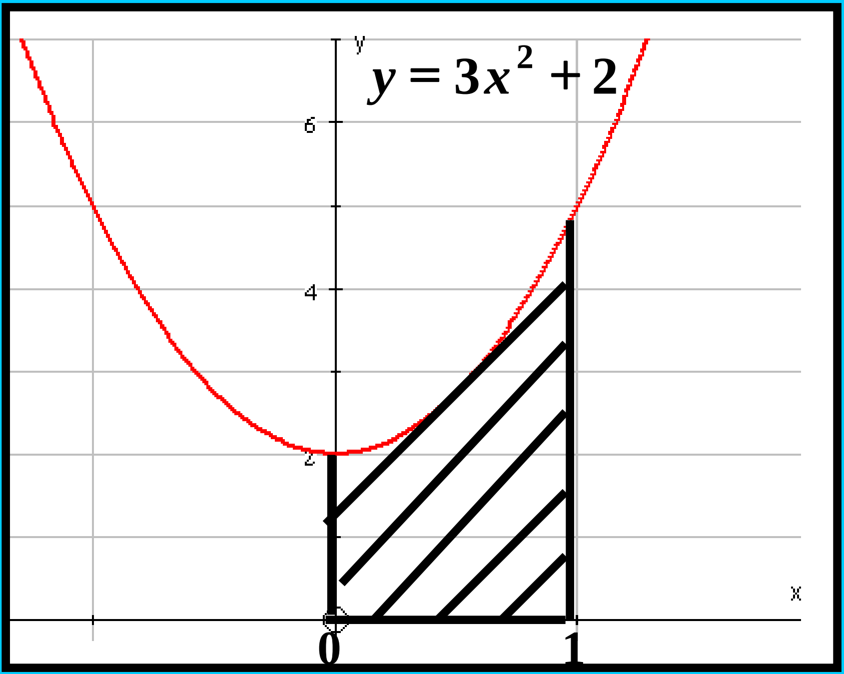
Definite Integration and Areas

Finding an area

Since $$\int_0^1 3x^2 + 2\,dx$$

$$= \left[\, x^3 + 2x \,\right]_0^1 = 3$$

the shaded area equals 3



The units are usually unknown in this type of question

Definite Integration and Areas

SUMMARY

- The definite integral $\int_a^b f(x)\,dx$ or $\int_a^b y\,dx$

gives the area between

- the curve $y = f(x)$,
- the x-axis and
- the lines $x = a$ and $x = b$

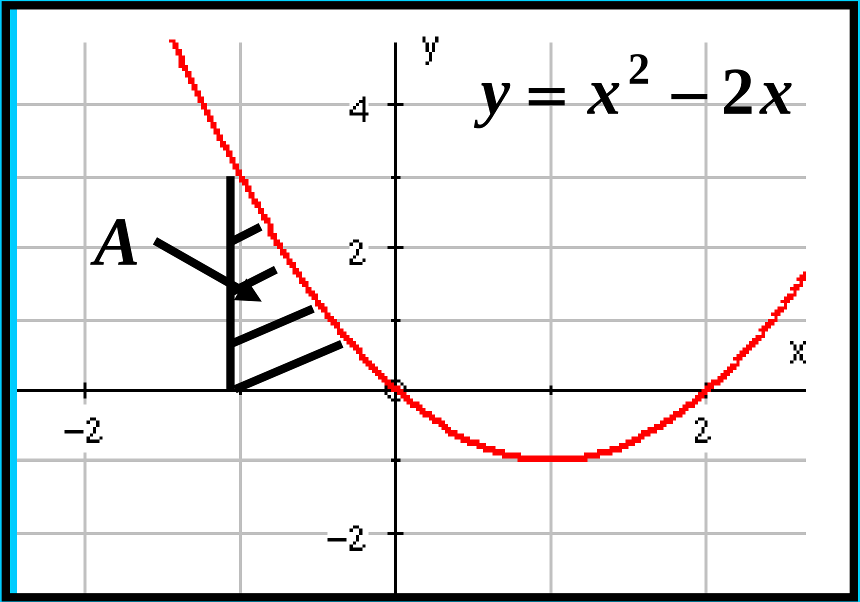
PROVIDED that

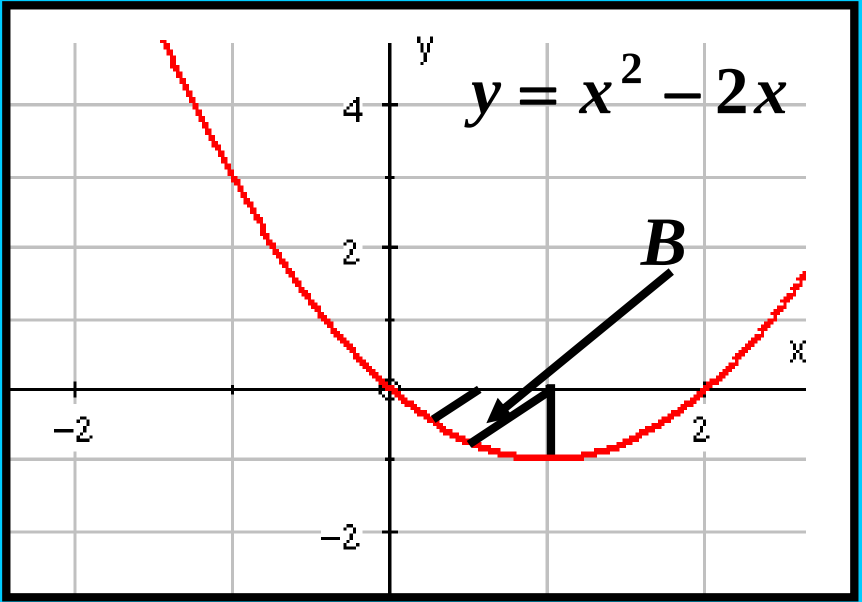
the curve lies on, or above, the x-axis between

the values $x = a$ and $x = b$

Definite Integration and Areas

Finding an area





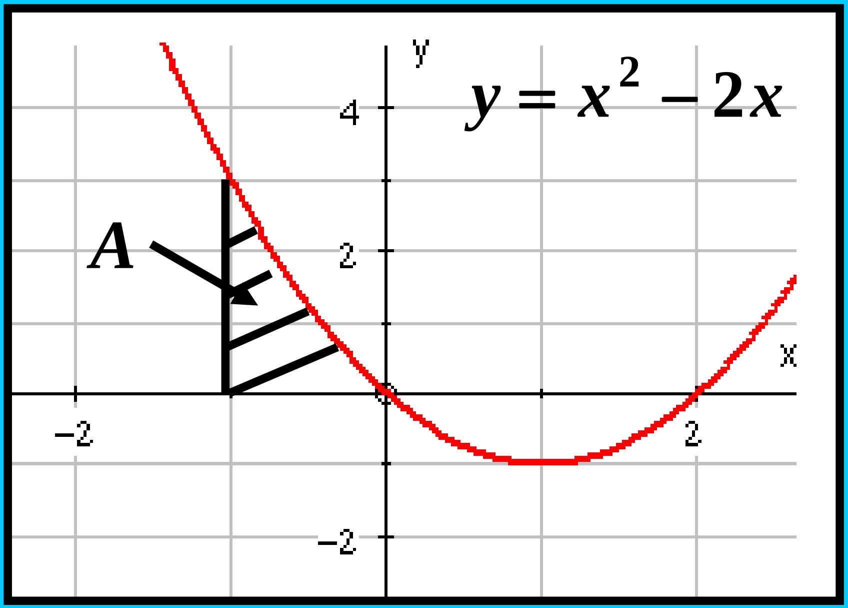
area $A = \int_{-1}^{0} x^2 - 2x \, dx$

For parts of the curve below the x-axis, the definite integral is negative, so

area $B = -\int_{0}^{1} x^2 - 2x \, dx$

Definite Integration and Areas

Finding an area



$$A = \int_{-1}^{0} x^2 - 2x \; dx$$

$$= \left[\frac{x^3}{3} - \frac{\overset{1}{\cancel{2}}x^2}{\underset{1}{\cancel{2}}} \right]_{-1}^{0}$$

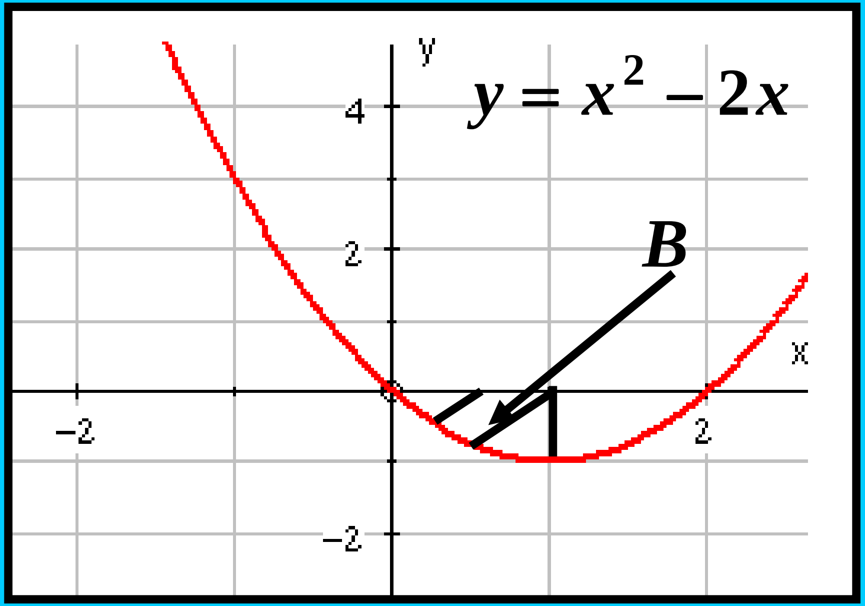
$$= \left[\; 0 \; \right] - \left[\frac{(-1)^3}{3} - (-1)^2 \right]$$

$$= -\left[-\frac{1}{3} - 1 \right]$$

$$\Rightarrow \quad \textbf{Area } A \quad = \frac{4}{3}$$

Definite Integration and Areas

Finding an area



$$-B = \int_0^1 x^2 - 2x \; dx$$

$$= \left[\frac{x^3}{3} - x^2 \right]_0^1$$

$$= \left[\frac{1}{3} - 1 \right] - \left[\; 0 \; \right]$$

$$= -\frac{2}{3}$$

$\Rightarrow$ Area $B = \dfrac{2}{3}$

Definite Integration and Areas

SUMMARY

- An area is always positive.
- The definite integral is positive for areas above the x-axis but negative for areas below the axis.
- To find an area, we need to know whether the curve crosses the x-axis between the boundaries.
  - For areas above the axis, the definite integral gives the area.
  - For areas below the axis, we need to change the sign of the definite integral to find the area.

Definite Integration and Areas

Exercise

Find the areas described in each question.

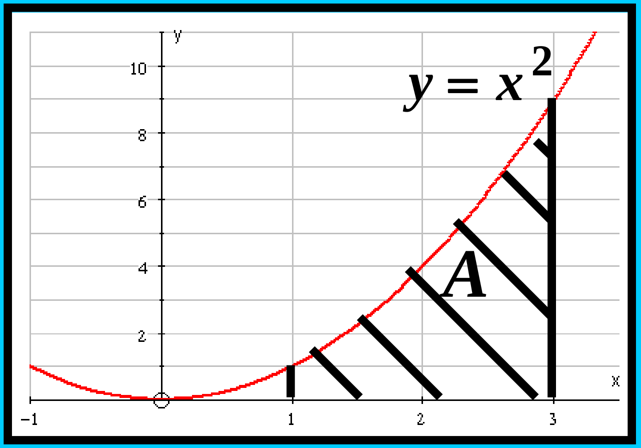
1. The area between the curve $y = x^2$ the x-axis and the lines $x = 1$ and $x = 3$.

2. The area between the curve $y = (x-1)(x-3)$, the x-axis and the $x = 2$ and $x = 3$.

Definite Integration and Areas

Solutions:

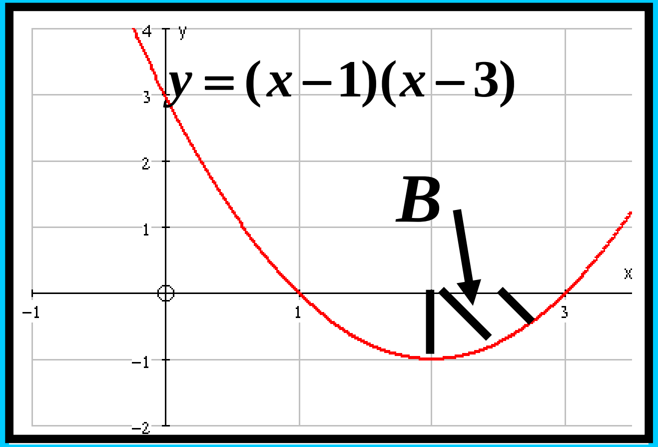
1.



$$A = \int_1^3 x^2 dx = \left[\frac{x^3}{3}\right]_1^3$$

$$= \left[\frac{(3)^3}{3}\right] - \left[\frac{(1)^3}{3}\right] = 8\tfrac{2}{3}$$

2.



$$-B = \int_2^3 x^2 - 4x + 3\,dx$$

$$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_2^3 = -\frac{2}{3}$$

$$\Rightarrow B = \frac{2}{3}$$

Definite Integration and Areas

Extension

The area bounded by a curve, the **y-axis** and the lines $y = c$ and $y = d$ is found by switching the xs and ys in the formula.

So, $\int_a^b y \, dx$ becomes $\int_c^d x \, dy$

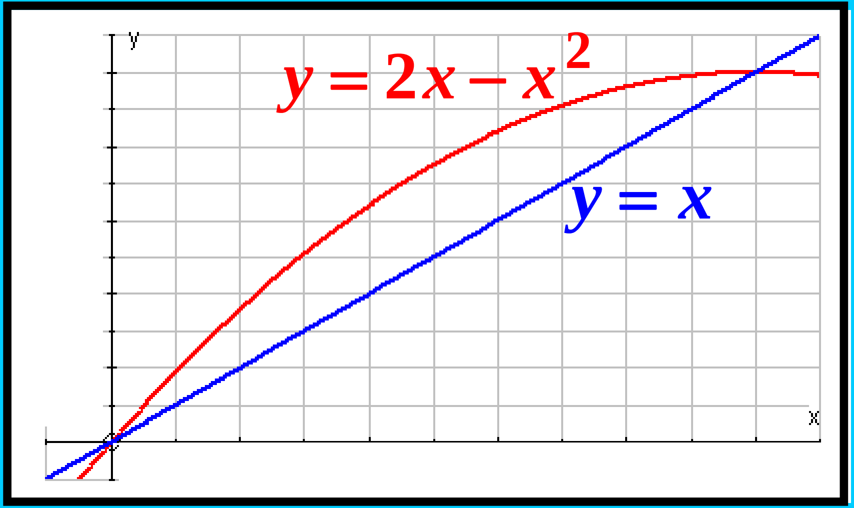
e.g. To find the area between the curve $y = \sqrt{x}$, the y-axis and the lines $y = 1$ and $y = 2$, we need

$$\int_c^d x \, dy = \int_1^2 y^2 \, dy = \frac{7}{3}$$

Definite Integration and Areas

Harder Areas

e.g.1 Find the coordinates of the points of intersection of the curve and line shown. Find the area enclosed by the curve and line.



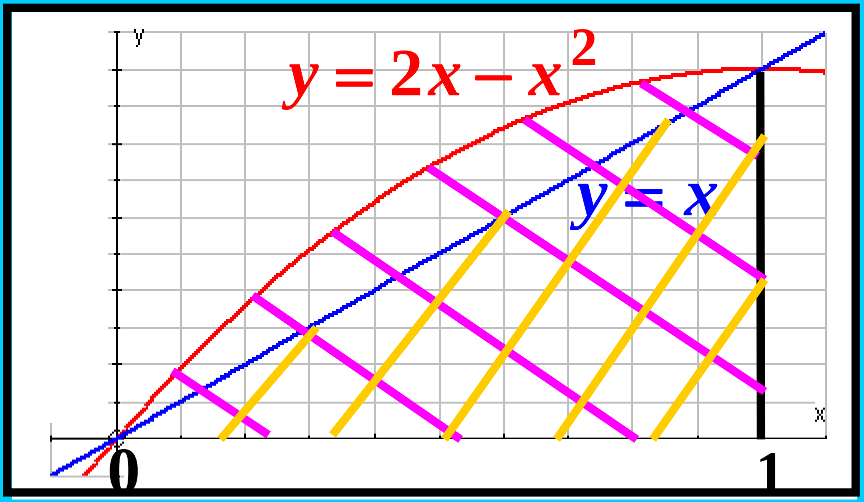
Solution: The points of intersection are given by

$$x = 2x - x^2$$

$\Rightarrow \quad x^2 - x = 0 \quad \Rightarrow \quad x(x-1) = 0$

$\Rightarrow \quad x = 0$ or $x = 1$

Definite Integration and Areas



Substitute in $y = x$

$$x = 0 \quad \Rightarrow \quad y = 0$$

$$x = 1 \quad \Rightarrow \quad y = 1$$

The area required is the area under the curve between 0 and 1 . . .

. . . minus the area under the line (a triangle)

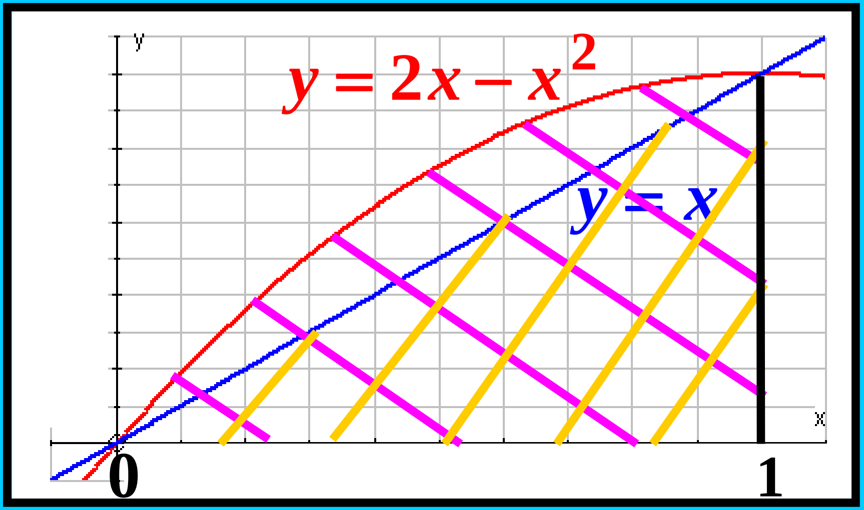
Method 1

Area under the curve $= \displaystyle\int_0^1 2x - x^2 dx = \left[x^2 - \frac{x^3}{3} \right]_0^1 = \frac{2}{3}$

Area of the triangle $= \dfrac{1}{2}(1)(1) = \dfrac{1}{2}$

$\Rightarrow$ Required area $= \dfrac{2}{3} - \dfrac{1}{2} = \dfrac{1}{6}$

Definite Integration and Areas



Method 2

Instead of finding the 2 areas and then subtracting, we can subtract the functions before doing the integration.

We get $2x - x^2 - x$

$= x - x^2$

Area $= \int_0^1 x - x^2 dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$

$= \left[\frac{1}{2} - \frac{1}{3} \right] - \left[0 \right] = \frac{1}{6}$

Definite Integration and Areas

Exercise

- Find the points of intersection of the following curves and lines. Show the graphs in a sketch, shade the region bounded by the graphs and find its area.

(a) $y = x^2 + 2$; $y = 6$ (b) $y = 4 - x^2$; $y = x + 2$

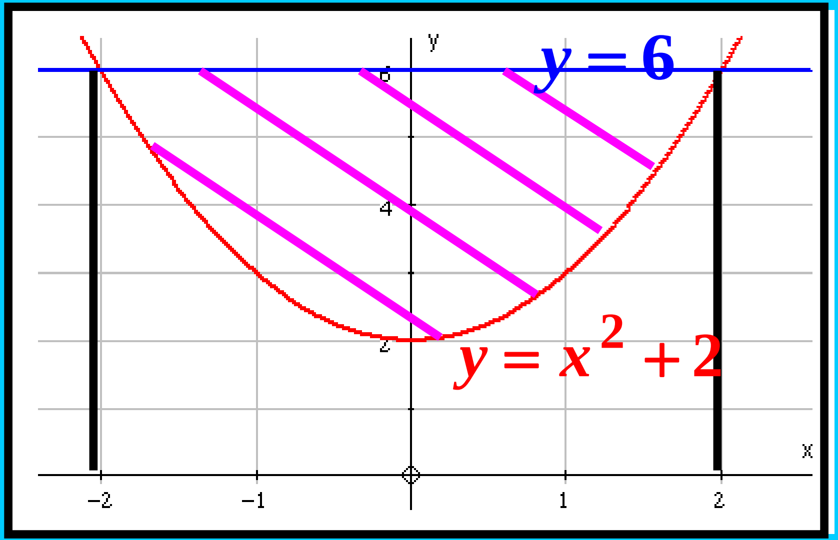
Solution:

(a) $x^2 + 2 = 6$

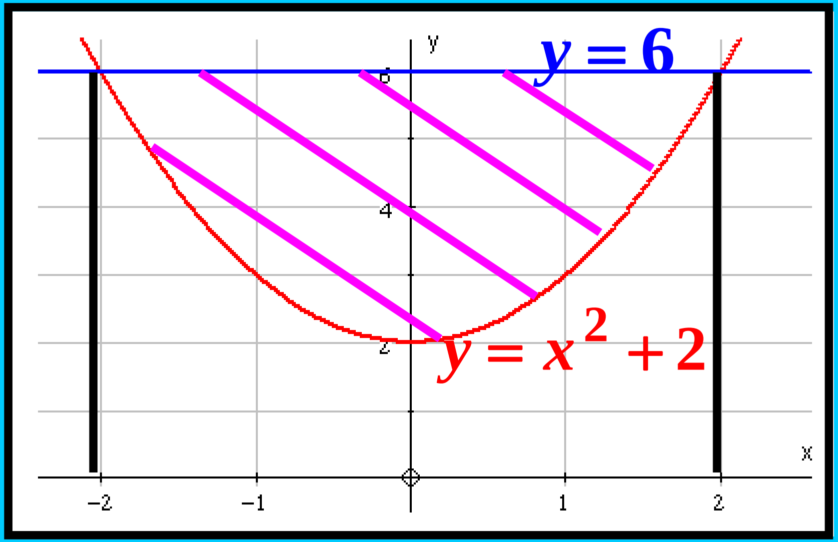
$\Rightarrow \quad x^2 = 4$

$\Rightarrow \quad x = \pm 2$

($y = 6$ for both points)



Definite Integration and Areas



Shaded area = area of rectangle – area under curve

Area under curve $= \int_{-2}^{2} x^2 + 2\,dx = \left[\frac{x^3}{3} + 2x\right]_{-2}^{2}$

$$= \left[\frac{8}{3} + 4\right] - \left[\frac{-8}{3} + 4\right] = \frac{16}{3}$$

Shaded area $= 24 - \frac{16}{3} = 18\frac{2}{3}$

Definite Integration and Areas

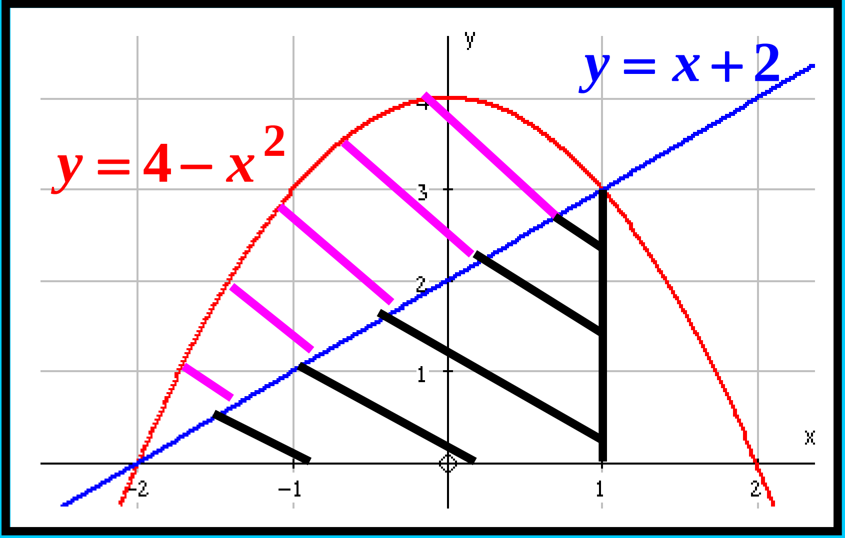
(b) $y = 4 - x^2$; $y = x + 2$

$\Rightarrow \quad x + 2 = 4 - x^2$

$\Rightarrow \quad x^2 + x - 2 = 0$

$\Rightarrow \quad (x+2)(x-1) = 0$

$\Rightarrow \quad x = -2$ or $x = 1$



Substitute in $y = x + 2$:

$x = -2 \Rightarrow y = 0, \quad x = 1 \Rightarrow y = 3$

Area under the curve $= \displaystyle\int_{-2}^{1} 4 - x^2 \, dx = \left[4x - \frac{x^3}{3} \right]_{-2}^{1} = 9$

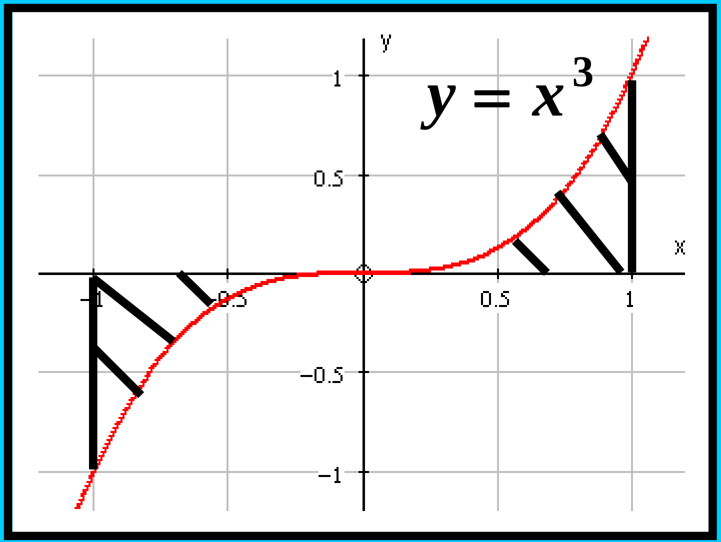
Area of the triangle $= \frac{1}{2} \times 3 \times 3 = \frac{9}{2}$

Shaded area = area under curve - area of triangle $= \frac{9}{2}$

Definite Integration and Areas

If a curve crosses the x-axis between the limits of integration, part of the area will be above the axis and part below.

e.g. $y = x^3$ between -1 and $+1$



The symmetry of the curve means that the integral from -1 to $+1$ is 0.

To find the area, we could integrate from 0 to 1 and, because of the symmetry, double the answer.

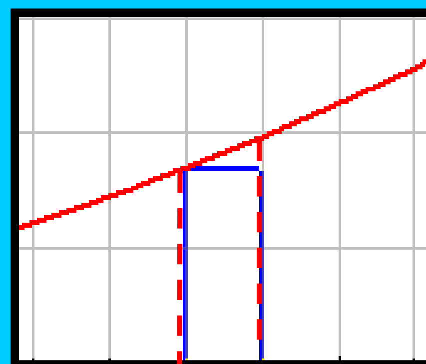
For a curve which wasn't symmetrical, we could find the 2 areas separately and then add.

Definite Integration and Areas

You don't need to know how the formula for area using integration was arrived at, but you do need to know the general ideas.

The area under the curve is split into strips.

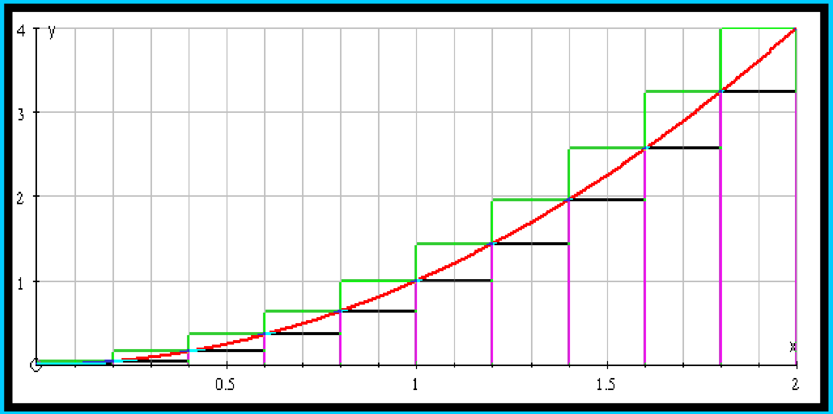
The area of each strip is then approximated by 2 rectangles, one above and one below the curve as shown.



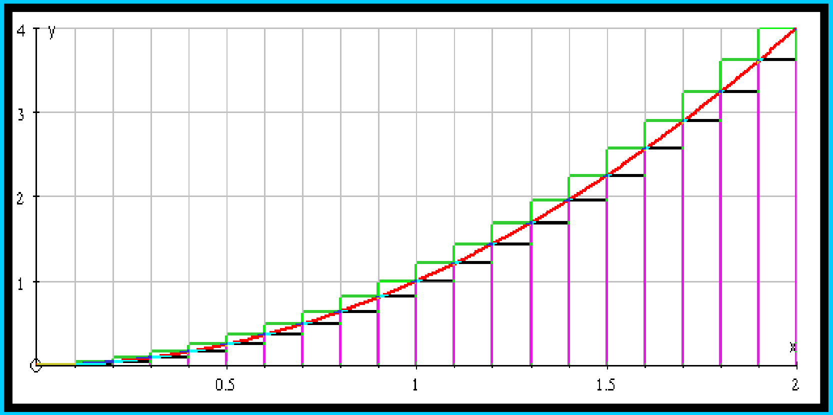
The **exact** area of the strip under the curve lies between the area of the 2 rectangles.

Definite Integration and Areas

Using 10 rectangles below and 10 above to estimate an area below a curve, we have . . .



Greater accuracy would be given with 20 rectangles below and above . . .



For an exact answer we let the number of rectangles approach infinity.

The exact area is "squashed" between 2 values which approach each other. These values become the definite integral.

Definite Integration and Areas

Definite Integration and Areas

The following slides contain repeats of information on earlier slides, shown without colour, so that they can be printed and photocopied.

For most purposes the slides can be printed as "Handouts" with up to 6 slides per sheet.

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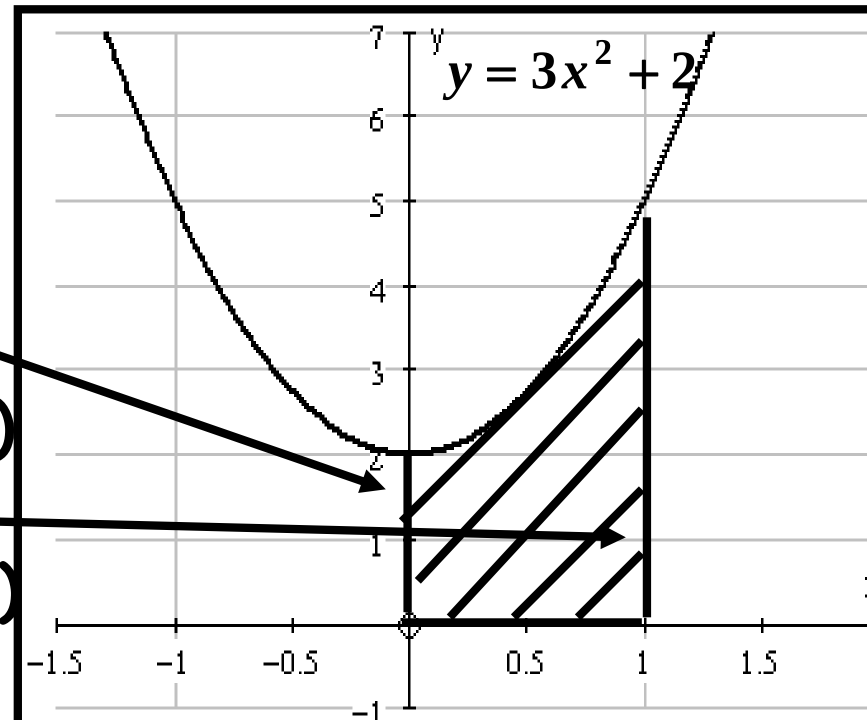

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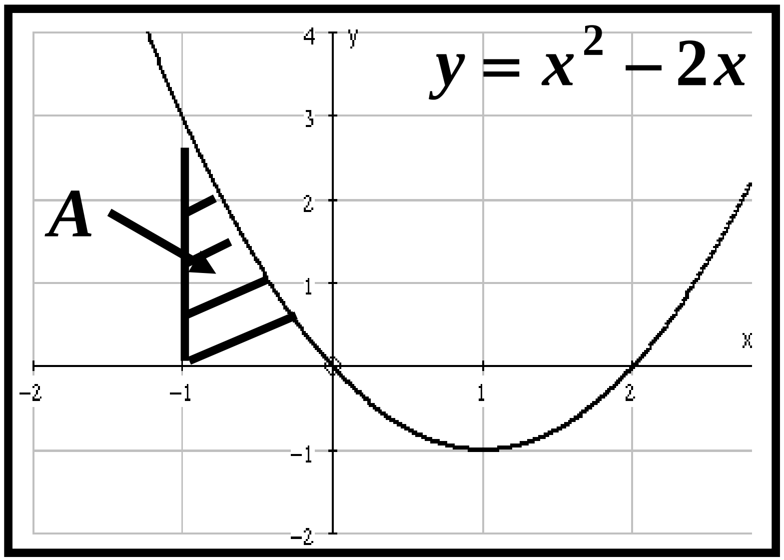

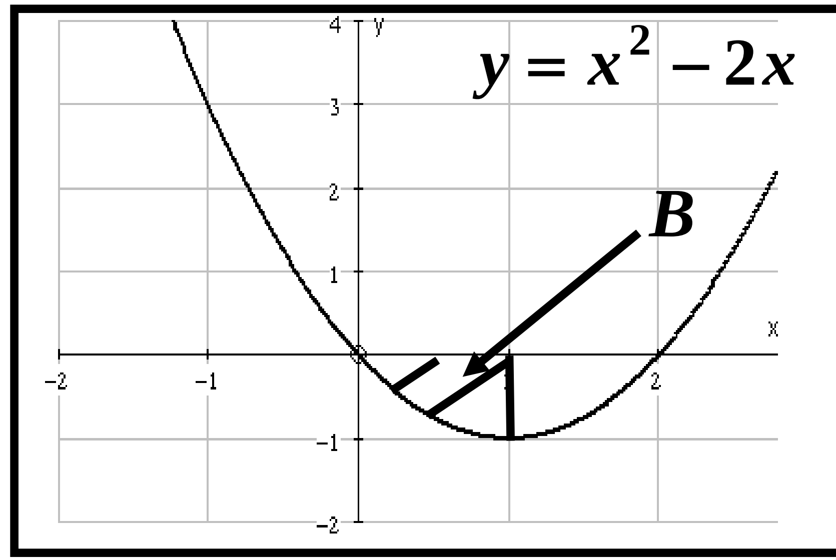

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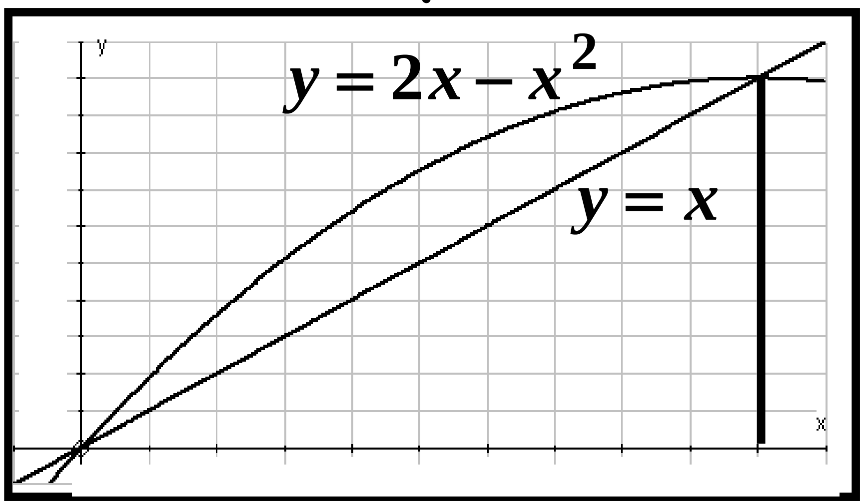

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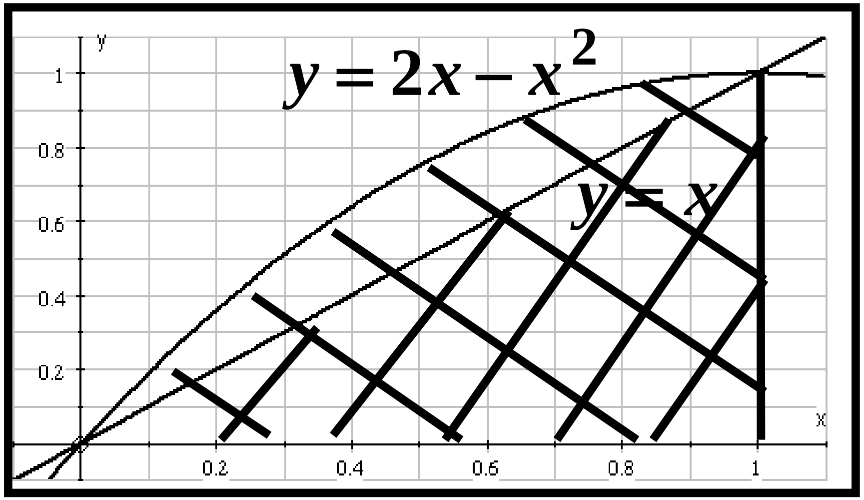

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